198. Proposed by ARTEMAS MARTIN, Washington, D. C.

Prove that every even number is the sum of two prime numbers.

Note.—This problem has long been known and no proof has ever been given. [EDITORS.]

CORRECTION.—Number **192**, incorrectly given as **188** in the June issue, page 196, should have the zeros on the right of equations (1), (2), (3) each replaced by $\square$, the symbol for a "square number."

SOLUTIONS OF PROBLEMS.

ALGEBRA.

386. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Given the sequence, $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \cdots, u_1, u_2, u_3, \cdots$, show that

$$\left.\frac{\sum_1^n u_i}{n}\right]_{n \doteq \infty} = \frac{1}{2}.$$

SOLUTION BY H. A. LEVY, Houghton, Mich.

Consider the sequence $\frac{1}{q}, \frac{2}{q}, \frac{3}{q}, \cdots, \frac{q-1}{q}$.

(1) Suppose q to be an odd prime. Then the quantities $\frac{n}{q}$ $(n = 1, 2, 3, \cdots, q-1)$ are irreducible fractions, and $q-1$ must be an even number, say $2k$. Then

$$\sum_{n=1}^{q-1} \frac{n}{q} = \frac{1+2+3+\cdots+(q-1)}{q} = \frac{q-1}{2} = k.$$

Hence, in such a sequence, there are $2k$ terms, and the sum of the terms is k.

(2) Suppose that q is an even number of the form $2^r p$, where p is any odd prime except one. All the fractions with even numerators are then reducible, and the sequence is

$$\frac{1}{2^r p}, \frac{3}{2^r p}, \frac{5}{2^r p}, \cdots, \frac{2^r p - 1}{2^r p}.$$

In this sequence, at least one term is reducible, viz., $\frac{p}{2^r p}$. If there are more, they are the terms $\frac{p}{2^r p}, \frac{3p}{2^r p}, \frac{5p}{2^r p}, \cdots$ up to the greatest odd multiple of p less than $2^r p$. This odd multiple must therefore be of the form $4k+1$. Hence an odd number of terms are reducible. Furthermore, the numerator of the last term of the sequence, $2^r p - 1$, is of the form $4k+1$, so that there is an odd number of terms in the sequence. Rejecting the reducible terms, therefore, leaves an even number of irreducible terms.

If $r = 1$, the number of terms is $p - 1$, an even number. The sum of the terms is then

$$\left(\frac{1}{2p}+\frac{3}{2p}+\frac{5}{2p}+\cdots+\frac{2p-1}{2p}\right)-\frac{p}{2p}=\frac{p}{2}-\frac{1}{2}=\frac{p-1}{2}.$$

That is, the sum of the terms is half the number of terms. The extension of the proof for the case where $r > 1$ is obvious.

(3) Suppose q is the product of two primes, say $l \cdot m$. If either l or m is 2, this reduces to the case just considered. The sequence is now

$$\frac{1}{lm},\ \frac{2}{lm},\ \frac{3}{lm},\ \cdots,\ \frac{lm-1}{lm},$$

an even number of terms, since lm is odd. An even number of terms are reducible. They are

$$\frac{l}{lm},\ \frac{2l}{lm},\ \frac{3l}{lm},\ \cdots,\ \frac{kl}{lm},\ \frac{m}{lm},\ \frac{2m}{lm},\ \frac{3m}{lm},\ \cdots,\ \frac{k'l}{lm}.$$

k and k' are both even, because k is the greatest multiple of l less than m and k' is the greatest multiple of m less than l. This leaves an even number of irreducible fractions in the sequence.

The sequence then is

$$\frac{1}{lm}+\frac{2}{lm}+\frac{3}{lm}+\cdots+\frac{lm-1}{lm}$$
$$-\left[\frac{1}{m}+\frac{2}{m}+\frac{3}{m}+\cdots+\frac{m-1}{m}+\frac{1}{l}+\frac{2}{l}+\frac{3}{l}+\cdots+\frac{l-1}{l}\right].$$

There are $lm - l - m + 1$ terms in the sequence, and the sum of the terms is

$$\frac{lm-l-m+1}{2}.$$

(4) Hence in all cases, the number of irreducible fractions having the same denominator is even, and their sum is half the number of terms.

(5) There remains to be considered one exception, the fractions having 2 as a denominator. Here there is only one term, but the sum of the terms is still one half the number of terms.

(6) Associating together the sequences having 2 as a denominator, 3 as a denominator, 4 as a denominator, etc., we have, besides the term $\frac{1}{2}$,

a sequence of $2k'$ terms whose sum is k',

a sequence of $2k''$ terms whose sum is k'', etc.

Adding these, we have

$1 + 2(k' + k'' + k''' + \cdots)$ terms whose sum is $\frac{1}{2} + k' + k'' + k''' + \cdots$.

Therefore

$$\frac{\sum_1^m u_i}{n}=\frac{\frac{1}{2}+k'+k''+k'''+\cdots}{1+2(k'+k''+k'''+\cdots)}=\frac{1}{2}.$$

Also solved by H. C. Feemster.